W4. José Luis Díaz-Barrero

A sequence of integers $\{a_n\}_{n\geq 1}$ is given by the conditions $a_1 = 1, a_2 = 12, a_3 = 20$ and $a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n$ for every $n \geq 1$. Prove that for every positive integer n, the number

$1 + 4a_n a_{n+1}$ is a perfect square.

Solution by Arkady Alt , San Jose ,California, USA.

Since $a_{n+3} - 2a_{n+2} - 2a_{n+1} + a_n = a_{n+3} - 3a_{n+2} + a_{n+1} + a_{n+2} - 3a_{n+1} + a_n = 0 \Leftrightarrow$

$a_{n+3} - 3a_{n+2} + a_{n+1} = (-1)(a_{n+2} - 3a_{n+1} + a_n)$ then

$a_{n+2} - 3a_{n+1} + a_n = (-1)^{n-1}(a_3 - 3a_2 + a_1) = (-1)^{n-1}(20 - 36 + 1) = 15(-1)^n \Leftrightarrow$

$(-1)^{n+2}a_{n+2} + 3(-1)^{n+1}a_{n+1} + (-1)^n a_n = 15 \Leftrightarrow$

$\left((-1)^{n+2}a_{n+2} - 3\right) + 3\left((-1)^{n+1}a_{n+1} - 3\right) + ((-1)^n a_n - 3) = 0.$

Denoting $b_n := (-1)^n a_n - 3$ we obtain homogeneos linear recurrence of the second degree $b_{n+2} + 3b_{n+1} + b_n = 0, n \in \mathbb{N}$ with initial conditions $b_1 = -4, b_2 = 9$ and since $a_n = (-1)^n(b_n + 3)$ then

$1 + 4a_n a_{n+1} = 1 - 4(b_n + 3)(b_{n+1} + 3) = -35 - 12b_n - 4b_n b_{n+1} - 12b_{n+1}.$

Noting that $b_0 = -3b_1 - b_2 = 3$ and $b_{n+1}b_{n-1} - b_n^2 =$

$(-3b_n - b_{n-1})b_{n-1} - (-3b_{n-1} - b_{n-2})b_n = b_n b_{n-2} - b_{n-1}^2, n \geq 2$ we can conclude that

$b_{n+1}b_{n-1} - b_n^2 = b_2 b_0 - b_1^2 = 11.$

Since $b_{n-1}(b_{n+1} + 3b_n + b_{n-1}) = 0 \Leftrightarrow b_{n-1}b_{n+1} + 3b_n b_{n-1} + b_{n-1}^2 = 0 \Leftrightarrow$

$3b_n b_{n-1} = -b_{n-1}b_{n+1} - b_{n-1}^2$ and $b_{n+1}b_{n-1} = b_n^2 + 11$ then

$3b_n b_{n-1} = -b_{n-1}^2 - b_n^2 - 11, n \geq 1 \Leftrightarrow 3b_{n+1}b_n = -b_n^2 - b_{n+1}^2 - 11, n \geq 0.$

Hence, $1 + 4a_n a_{n+1} = -35 - 12b_n - 12b_{n+1} + 8b_n b_{n+1} - 12b_n b_{n+1} =$

$-35 - 12b_n - 12b_{n+1} + 8b_n b_{n+1} + 44 + 4b_{n+1}^2 + 4b_n^2 =$

$4b_{n+1}^2 + 4b_n^2 + 9 - 12b_n - 12b_{n+1} + 8b_n b_{n+1} = (2b_n + 2b_{n+1} - 3)^2.$

Remark.

Let $t_n := 3 - 2b_{n+1} - 2b_n,$ then $1 + 4a_n a_{n+1} = t_n^2$ where t_n satisfy to the recurrence $3 - t_{n+1} + 3(3 - t_n) + 3 - t_{n-1} = 0 \Leftrightarrow$

$$t_{n+1} + 3t_n + t_{n-1} = 15$$

and $t_0 = 3 - 2c_0 - 2c_1 = 5, t_1 = 3 - 2c_1 - 2c_2 = -7.$